

Supersymmetry of the Schrödinger and PP Wave Solutions in Einstein-Weyl Supergravities

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ABSTRACT

We obtain the Schrödinger and general pp-wave solutions with or without the massive vector in Einstein-Weyl supergravity. The vector is an auxiliary field in the off-shell supermultiplet and it acquires a kinetic term in the Weyl-squared super invariant. We study the supersymmetry of these solutions and find that turning on the massive vector has a consequence of breaking all the supersymmetry. The Schrödinger and also the pp-wave solutions with the massive vector turned off on the other hand preserve $\frac{1}{4}$ of the supersymmetry.

1 Introduction

One important application of the AdS/CFT correspondence involves the construction of gravity duals for non-relativistic field theories. The non-relativistic conformal symmetry is known as Schrödinger symmetry, which is a symmetry group of the Schrödinger equation for free fermions [1, 2] or of fermions at unitarity [3]. Recently, it was proposed that the corresponding gravity background is a special class of the cosmological pp-wave solution, namely [4, 5]

$$ds^2 = \ell^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 (-2dt dx + dy^i dy^i) \right). \quad (1)$$

For $z = 1$, it is simply the AdS vacuum with the full Lorentzian conformal symmetry. For $z = -1/2$, the metric is also Einstein, known as Kaigorodv metric, describing a pp-wave propagating in the AdS background. It was demonstrated that the isometry of this metric has the Schrödinger symmetry when $z = 2$ [4]. For general z , one special conformal transformation is lost [5]. The metric is homogeneous with the following scaling symmetry

$$r \rightarrow \lambda^{-1} r, \quad t \rightarrow \lambda^z t, \quad x \rightarrow \lambda^{2-z} x, \quad y \rightarrow \lambda y. \quad (2)$$

The Schrödinger solution belongs to the general class of pp-wave solutions. For a generic scaling exponent z , the solution can be obtained by introducing a massive vector field in Einstein gravity [4]. An alternative and well studied geometry that breaks the Lorentz symmetry is the Lifshitz solutions [6]. (See also [7] for an earlier discussion.)

Massive vector fields are typically absent in usual on-shell supergravities. However, they can arise in supergravities in the off-shell formalism as auxiliary fields in the off-shell supermultiplets. They are called auxiliary fields because in a two-derivative off-shell supergravity the equations of motion of these fields are algebraic. However, they can become dynamical when additional higher-order super invariants are added in the Lagrangian. The simplest such a theory is perhaps the Einstein-Weyl supergravity in four dimensions [8, 9]. It is the supersymmetric generalization of the Einstein-Weyl gravity which has been recently studied for its critical behavior [10]. Schrödinger and more general pp-wave solutions were demonstrated to exist in Einstein-Weyl or more general higher-derivative gravities [11, 12, 13]. Furthermore, it was shown that there exist new asymptotic AdS and Lifshitz black holes in Einstein-Weyl gravity [14]. Supersymmetric AdS and Lifshitz solutions were also constructed recently, in which the massive vector plays an important role for the solutions to be supersymmetric [15]. These examples show that higher-derivative gravities and supergravities have rich structures for constructing geometric backgrounds that are dual to non-relativistic field theories, as well as the relativistic ones.

In this paper, we construct the Schrödinger and general pp-wave solutions in Einstein-Weyl supergravity. Turning on the massive vector gives rise to new such solutions. We then study the supersymmetry of these solutions. We find that the pure gravitational pp-wave solutions always preserve $\frac{1}{4}$ of the supersymmetry. Turning on the massive vector in these solutions always breaks the supersymmetry. In contrast, for supersymmetric Lifshitz solutions in Einstein-Weyl supergravity, the supersymmetry requires a non-vanishing massive vector [15].

The paper is organized as follows. In section 2, we give a quick review of Einstein-Weyl off-shell supergravity. In section 3, we construct Schrödinger solutions with or without the massive vector. We then study their supersymmetry. In section 4, we construct a general class of pp-wave solutions and then study the supersymmetry. In particular we obtain solutions that describe flows from one Schrödinger solution to another with different scaling symmetries (2). We conclude our paper in section 5.

2 Einstein-Weyl supergravity

The field content of the off-shell $\mathcal{N} = 1$, $D = 4$ supergravity consists of the vielbein e_μ^a , a massive vector A and a complex scalar $S + iP$, totalling 12 off-shell degrees of freedom, matching with that of the gravitino ψ_μ . The general formalism for constructing a supersymmetric action for any chiral superfield was obtained in [16]. For appropriate choices of superfields, one obtains the actions of the supersymmetrizations of the cosmological term, the Einstein Hilbert term and the Weyl-squared terms [8]. Adopting the notation of [9], the bosonic Lagrangian is given by

$$e^{-1}\mathcal{L} = R + \frac{2}{3}(A^2 - S^2 - P^2) + 4S\sqrt{-\Lambda/3} + \frac{1}{2}\alpha C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} - \frac{1}{3}\alpha F^2, \quad (3)$$

where C is the Weyl tensor and $F = dA$. The supersymmetric transformation rule for the gravitino is given by

$$\delta\psi_\mu = -D_\mu\epsilon - \frac{i}{6}(2A_\mu - \Gamma_{\mu\nu}A^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5P)\epsilon. \quad (4)$$

The leading-order supergravity with $\alpha = 0$ (and also $\Lambda = 0$) was constructed much earlier in [17, 18]. The super invariant associated with the cosmological term $4S\sqrt{-\Lambda/3}$ was introduced in [9].

The equations of motion for the scalar fields S and P imply that

$$S = 3\sqrt{-\Lambda/3}, \quad P = 0, \quad (5)$$

and hence they are auxiliary with no dynamical degree of freedom. The equations of motion for the vector field and the metric are given by

$$\begin{aligned} 0 &= \alpha \nabla^\mu F_{\mu\nu} + A_\nu, \\ 0 &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{2}{3} \alpha (F_{\mu\nu}^2 - \frac{1}{4} F^2 g_{\mu\nu}) \\ &\quad + \frac{2}{3} (A_\mu A_\nu - \frac{1}{2} A^2 g_{\mu\nu}) - \alpha (2 \nabla^\rho \nabla^\sigma + R^{\rho\sigma}) C_{\mu\rho\sigma\nu}. \end{aligned} \quad (6)$$

Thus we see that for $\alpha = 0$, the bosonic theory is effectively Einstein gravity with a cosmological term, with the vector set to zero by the equations of motion. On the other hand, if α is infinity, the theory is conformal gravity with the vector becoming a Maxwell field in the bosonic sector. It was shown in [9] that the theory admits a supersymmetric AdS vacuum with the cosmological constant Λ . The linear spectrum on the AdS background was analyzed. The gravity modes are identical to those in Einstein-Weyl gravity studied in [10]. (See also [19, 20, 21, 22].) There is a ghost massive spin-2 mode in addition to the massless graviton. The mass is determined by the quantity $\alpha\Lambda$, the product of the cosmological constant and the coupling of the Weyl-squared term. There is a critical phenomenon when

$$\alpha\Lambda = \frac{3}{2}, \quad (7)$$

for which the massive spin-2 mode disappears and is replaced by the log mode [10, 11, 20]. Recently, supersymmetric asymptotic AdS and Lifshitz solutions were obtained in [15].

3 Supersymmetry of the Schrödinger solutions

3.1 Schrödinger solutions

Let us consider the following ansatz for the Schrödinger solutions [4, 5]

$$ds^2 = \ell^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 (-2dxdt + dy^2) \right), \quad A = q r^z dt, \quad (8)$$

where q is a constant. Note that the scaling symmetry (2) is preserved even though the massive vector field is turned on. We find that the equation for the massive vector implies that

$$q(\ell^2 + \alpha z(z+1)) = 0. \quad (9)$$

Thus we see that for non-vanishing q , the coupling constant α is determined fully by the above solution, giving rise to the Schrödinger solution with

$$\Lambda = -\frac{3}{\ell^2}, \quad \alpha = -\frac{\ell^2}{z(z+1)}, \quad q = \frac{3(z-1)}{\sqrt{2}}. \quad (10)$$

On the other hand, if the massive vector field is turned off, namely $q = 0$, we have

$$\Lambda = -\frac{3}{\ell^2}, \quad \alpha = -\frac{\ell^2}{2z(2z-1)}. \quad (11)$$

When $z = 1$, the solution is the AdS vacuum; when $z = -1/2$, the solution is the Kaigrodov metric describing a pp-wave propagating in the AdS. Both solutions are Einstein and hence exist for all values of α .

The scaling exponent z is determined by the product of the cosmological constant and the coupling of the Weyl-squared super invariant term. For each $\alpha\Lambda$, there can be two allowed z , given by

$$\begin{aligned} q = 0 : \quad z_{\pm} &= \frac{1}{4} \pm \frac{1}{4} \sqrt{1 + \frac{12}{\alpha\Lambda}}, \\ q \neq 0 : \quad z_{\pm} &= -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{12}{\alpha\Lambda}}, \end{aligned} \quad (12)$$

It is of interest to note that at the criticality, both has the same $z = 1$ root. As we shall see later, there are more general solutions that describe flows from one Schrödinger solution with z_- to the other with z_+ .

3.2 The supersymmetry

We now investigate the supersymmetry of the Schrödinger solutions we have obtained so far. We do this by studying the Killing spinor equation $\delta\psi_\mu = 0$, namely

$$-D_\mu\epsilon - \frac{i}{6}(2A_\mu - \Gamma_{\mu\nu}A^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5P)\epsilon = 0. \quad (13)$$

To proceed, we choose the following vielbein base

$$e^+ = \ell r^z dt, \quad e^- = -\frac{1}{2}\ell r^z dt - \ell r^{2-z} dx, \quad e^{\bar{y}} = \ell r dy, \quad e^{\bar{r}} = \frac{\ell}{r} dr. \quad (14)$$

Here the indices $(+, -, \bar{y}, \bar{r})$ are those for the tangent spacetime. Note that the scaling symmetry (2) is preserved for each vielbein. The resulting spin connection has the following non-vanishing components

$$\omega^{\bar{y}\bar{r}} = \frac{1}{\ell}e^{\bar{y}}, \quad \omega^{+\bar{r}} = -\frac{1}{\ell}e^+, \quad \omega^{++} = -\frac{z-1}{\ell}e^{\bar{r}}, \quad \omega^{-\bar{r}} = \frac{1}{\ell}e^- - \frac{1}{\ell}(z-1)e^+. \quad (15)$$

We find that the components of the Killing spinor equation for the ansatz (8) become

$$\begin{aligned} \partial_x\epsilon - \frac{1}{2}r^{2-z}\Gamma_-(\Gamma_{\bar{r}}+1)\epsilon &= 0, \\ \partial_r\epsilon + \frac{1}{2r}(z-1)\Gamma_{+-}\epsilon + \frac{1}{2r}\Gamma_{\bar{r}}\epsilon &= \frac{i}{6r}q\Gamma_{\bar{r}}\Gamma_5\epsilon, \\ \partial_y\epsilon + \frac{1}{2}r\Gamma_{\bar{y}}(\Gamma_{\bar{r}}+1)\epsilon &= \frac{i}{6}qr\Gamma_{\bar{y}}\Gamma_5\epsilon, \end{aligned}$$

$$\partial_t \epsilon + \frac{1}{2} r^z \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon - \frac{1}{4} (2z - 1) r^z \Gamma_{-\bar{r}} \epsilon - \frac{1}{4} r^z \Gamma_- \epsilon = \frac{i}{6} q r^z (\Gamma_{+-} - 2) \Gamma_5 \epsilon, \quad (16)$$

where

$$\Gamma_+^2 = 0 = \Gamma_-^2, \quad \{\Gamma_+, \Gamma_-\} = 2. \quad (17)$$

For $q = 0$, we find that a Killing spinor exists, given by

$$\epsilon = r^{\frac{z}{2}} \epsilon_0, \quad \Gamma_- \epsilon_0 = 0, \quad \Gamma_{\bar{r}} \epsilon_0 = -\epsilon_0. \quad (18)$$

Thus the solution preserves $\frac{1}{4}$ of the supersymmetry. To show that there are no further Killing spinors, let us examine the integrability conditions. We find that

$$0 = [\partial_t, \partial_y] \epsilon = \frac{1}{2} r^{z+1} (z - 1) \Gamma_{\bar{y}-} \epsilon. \quad (19)$$

It follows that the Killing spinor must satisfy $\Gamma_- \epsilon = 0$ for $z \neq 1$. After imposing this condition, we find that

$$0 = [\partial_t, \partial_r] \epsilon = \frac{1}{2} r^{z-1} (z - 1) \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon. \quad (20)$$

Thus we see that for $z \neq 1$, the Killing spinor must satisfy the projections given in (18).

When q is non-vanishing, there is no Killing spinor at all and hence the solution (10) is not supersymmetric. This can be shown as follows. The integrability condition,

$$0 = [\partial_x, \partial_t] \epsilon = -\frac{i}{6} q r^2 (\Gamma_{\bar{r}} + 2) \Gamma_- \Gamma_5 \epsilon, \quad (21)$$

implies that any would-be Killing spinor must satisfy $\Gamma_- \epsilon = 0$. Imposing this condition, we find that

$$0 = [\partial_t, \partial_y] \epsilon = \frac{i}{2} q r^{z+1} \Gamma_{\bar{y}} \Gamma_5 \epsilon. \quad (22)$$

It is clear that there is no non-vanishing ϵ that satisfies the equation.

It is somewhat surprising that the pure gravitational Schrödinger solutions in Einstein-Weyl supergravity are supersymmetric whilst those with the non-vanishing massive vector are not. This is opposite to the Lifshitz solutions where the massive vector is indispensable for supersymmetry [15].

4 Supersymmetry of the PP-wave solutions

4.1 General PP wave solutions

The Schrödinger solutions belong to the general class of pp-wave solutions which we study in this section. The ansatz is given by

$$ds^2 = -\ell^2 \left(\frac{dr^2}{r^2} + r^2 (-H dt^2 - 2 dx dt + dy^2) \right),$$

$$A = \phi dt, \quad (23)$$

where H and ϕ are functions of the coordinates r, t and y . It becomes the Schrödinger solution if we have $H \sim \phi^2 \sim r^{2z}$. In general the solution is translational invariant only along the null x direction. Let $\Lambda = -3/\ell^2$. The full set of equations of motion reduces to

$$\alpha r^4 \phi_{,rr} + 2\alpha r^3 \phi_{,r} + \ell^2 r^2 \phi + \alpha \phi_{,yy} = 0. \quad (24)$$

$$\frac{r^2}{2\ell^2} \left(\alpha \square^2 + (2\alpha + \ell^2) \square \right) H = \frac{2}{3} \phi^2 - \frac{2\alpha}{3\ell^2 r^2} \left((\partial_y \phi)^2 + r^4 (\partial_r \phi)^2 \right), \quad (25)$$

where \square is the Laplacian of the AdS metric ($H = 0$) in (23):

$$\square = r^2 \partial_r^2 + 4r \partial_r + \frac{1}{r^2} \partial_y^2. \quad (26)$$

The critical case corresponds to having $\alpha = -\ell^2/2$ and the resulting equation of H involves only \square^2 .

Let us present the cohomogeneity-one solutions with H and ϕ being functions of r only. We find

$$\begin{aligned} \phi &= q_- r^{-\frac{1}{2}(1-m)} + q_+ r^{-\frac{1}{2}(1+m)}, \\ H &= c_0 + \frac{c_1}{r^3} + \frac{c_-}{m-3} r^{-\frac{1}{2}(3-m)} + \frac{c_+}{m+3} r^{-\frac{1}{2}(3+m)} \\ &\quad + \frac{8q_-^2}{9(3-m)^2} r^{-3+m} + \frac{8q_+^2}{9(3+m)^2} r^{-3-m}, \end{aligned} \quad (27)$$

where

$$m = \sqrt{1 - \frac{4\ell^2}{\alpha}}. \quad (28)$$

The solution is valid for general m except for $m^2 = 9$, corresponding to the critical point $\alpha = -\frac{1}{2}\ell^2$. At the critical point, the general solution is given by

$$\begin{aligned} \phi &= q_- r + \frac{q_+}{r^2}, \\ H &= c_0 + \frac{c_1}{r^3} + c_2 \log r + \frac{c_3(1+3\log r)}{r^3} + \frac{2q_+^2}{81r^6} + \frac{2}{27} q_-^2 (3\log r - 4) \log r. \end{aligned} \quad (29)$$

Note that since the equations of motion (24) and (25) do not involve a derivative with respect to t , it follows that the constant coefficients q_{\pm}, c_i can be arbitrary functions of t . The metric associated with the term c_1 , corresponding to $z = -\frac{1}{2}$, is the Kaigorodov metric and it is Einstein. It was conjectured that its boundary field theory is certain conformal field theory in the infinite momentum frame [23]. The coefficient c_0 can be removed by a shift of the coordinate x . The solution with $q_{\pm} = 0$ was obtained in [11]. It has a linear $\log r$ term, which is characteristic of $\square^2 \psi = 0$ in critical gravity. Interestingly, for non-vanishing q_- , we have a quadratic $\log r$ term, which is absent in pure Einstein-Weyl gravity.

In general, we can solve the vector equation (24) by separation of variables. Letting $\phi = \chi e^{iky}$, we find

$$\alpha r^4 \chi_{,rr} + 2\alpha r^3 \chi_{,r} + \ell^2 r^2 \chi - \alpha k^2 \chi = 0. \quad (30)$$

This equation can be solved in terms of Bessel functions

$$\phi = \frac{1}{\sqrt{r}} \left(\phi_1(t) I\left(-\frac{1}{2}m, \frac{k}{r}\right) + \phi_2 I\left(-\frac{1}{2}m, \frac{k}{r}\right) \right) e^{iky}. \quad (31)$$

Substituting this into (25) leads to the general equation for H . For $\phi = 0$, the solution was obtained in [11]. (See also [12, 13].)

4.2 From Schrödinger to Schrödinger

From the general cohomogeneity-one solution (27), we can obtain solutions that describe a flow from one Schrödinger vacuum to another when r runs from 0 to infinity. The scaling symmetry (2) breaks down for general r , but recovers at $r \rightarrow 0$ and $r \rightarrow \infty$, with the different scaling exponent at the two regions. For $q_+ = 0 = q_-$, we have

$$ds^2 = -(c_1 r^{2z_+} + c_2 r^{2z_-}) dt^2 + r^2 (-2dxdt + dy^2) + \frac{dr^2}{r^2}, \quad (32)$$

where

$$z_{\pm} = \frac{1}{4}(1 \pm m). \quad (33)$$

In addition, we can have metric that flows from the Kaigorodov (Schrödinger with $z = -1/2$) to a Schrödinger solution:

$$ds^2 = -\left(\frac{c_1}{r} + c_2 r^{2z}\right) dt^2 + r^2 (-2dxdt + dy^2) + \frac{dr^2}{r^2}, \quad (34)$$

where z can be either z_+ or z_- in (33). If we turn on A , the flow of the Schrödinger solutions is given by

$$ds^2 = -(\tilde{c}_1 r^{2z_+} + \tilde{c}_2 r^{2z_-}) dt^2 + r^2 (-2dxdt + dy^2) + \frac{dr^2}{r^2}, \quad A = (q_- r^{z_-} + q_+ r^{z_+}) dt, \quad (35)$$

where \tilde{c}_1, \tilde{c}_2 can be read off from (27) and

$$z_{\pm} = -\frac{1}{2}(1 \pm m). \quad (36)$$

Note that we cannot flow from pure gravitational Schrödinger solution to that with non-vanishing A .

4.3 The supersymmetry

To study the supersymmetry of the pp-wave solutions, we choose the following vielbein base for the metric (23)

$$e^+ = \ell r dt, \quad e^{\bar{y}} = \ell r dy, \quad e^{\bar{r}} = \frac{\ell}{r} dr, \quad e^- = -\frac{1}{2} \ell r H dt - \ell r dx. \quad (37)$$

The corresponding spin connection is given by

$$\begin{aligned} \omega^{\bar{y}\bar{r}} &= \frac{1}{\ell} e^{\bar{y}}, & \omega^{+\bar{r}} &= \frac{1}{\ell} e^+, & \omega^{-\bar{y}} &= -\frac{1}{2\ell r} \frac{\partial H}{\partial y} e^+, \\ \omega^{-\bar{r}} &= -\frac{1}{2\ell} r \frac{\partial H}{\partial r} e^+ + \frac{1}{\ell} e^-. \end{aligned} \quad (38)$$

Turning off A , the Killing spinor equations are given by

$$\begin{aligned} \partial_r \epsilon + \frac{1}{2r} \Gamma_{\bar{r}} \epsilon &= 0, & \partial_y \epsilon + \frac{1}{2} r \Gamma_{\bar{y}} (\Gamma_{\bar{r}} + 1) \epsilon &= 0, & \partial_x \epsilon - \frac{1}{2} r \Gamma_- (\Gamma_{\bar{r}} + 1) \epsilon &= 0, \\ \partial_t \epsilon + \frac{r}{2} \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon - \frac{1}{4} r H \Gamma_- (\Gamma_{\bar{r}} + 1) \epsilon - \frac{1}{4} \frac{\partial H}{\partial y} \Gamma_{-\bar{y}} \epsilon - \frac{1}{4} r^2 \frac{\partial H}{\partial r} \Gamma_{-\bar{r}} \epsilon &= 0. \end{aligned} \quad (39)$$

It is thus clear that a Killing spinor exists provided that

$$\epsilon = r^{\frac{1}{2}} \epsilon_0, \quad \Gamma_- \epsilon_0 = 0, \quad \Gamma_{\bar{r}} \epsilon_0 = -\epsilon_0. \quad (40)$$

The question remains whether this is the most general Killing spinor. To see this, let us examine the integrability conditions. We find that

$$[\partial_t, \partial_y] \epsilon = \frac{1}{4} \left((r^3 \partial_r H + \partial_y^2 H) \Gamma_{\bar{y}} + r^2 \partial_y \partial_r H \Gamma_{\bar{r}} \right) \Gamma_- \epsilon = 0. \quad (41)$$

This implies that in general we have $\Gamma_- \epsilon = 0$. (The possibility that the terms in bracket vanish will be discussed later.) This implies from (39) that

$$\partial_t \epsilon + \frac{1}{2} r \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon = 0 \quad (42)$$

This equation can be easily solved, giving $\epsilon = \epsilon(r, y) - \frac{1}{2} r t \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon(r, y)$. Imposing $\Gamma_- \epsilon = 0$ then leads to (40).

If we consider turning on the massive vector, *i.e.* $\phi(r, y, t)$ is non-vanishing, we find that the Killing spinor equations are given by,

$$\begin{aligned} \partial_r \epsilon - \frac{i}{6r^2} \phi \Gamma_{\bar{r}} \Gamma_5 \epsilon + \frac{1}{2r} \Gamma_{\bar{r}} \epsilon &= 0, \\ \partial_y \epsilon + \frac{1}{2} r \Gamma_{\bar{y}} (\Gamma_{\bar{r}} + 1) \epsilon - \frac{i}{6} \phi \Gamma_{\bar{y}} \Gamma_5 \epsilon &= 0, \\ \partial_x \epsilon - \frac{1}{2} r \Gamma_- (\Gamma_{\bar{r}} + 1) \epsilon &= 0, \\ \partial_t \epsilon + \frac{1}{2} r \Gamma_+ (\Gamma_{\bar{r}} + 1) \epsilon - \frac{1}{4} r H \Gamma_- (\Gamma_{\bar{r}} + 1) \epsilon - \frac{1}{4} \frac{\partial H}{\partial y} \Gamma_{-\bar{y}} \epsilon - \frac{1}{4} r^2 \frac{\partial H}{\partial r} \Gamma_{-\bar{r}} \epsilon &= 0. \end{aligned}$$

$$+\frac{i}{3}\phi\Gamma_5\epsilon - \frac{i}{6}\phi\Gamma_{+-}\Gamma_5\epsilon = 0. \quad (43)$$

If we impose the condition $\Gamma_-\epsilon = 0$, we find that the following integrability condition

$$0 = [\partial_y, \partial_t]\epsilon = -\frac{i}{2}r\phi\Gamma_y\Gamma_5\epsilon - \frac{i}{2}\partial_y\phi\Gamma_5\epsilon \quad (44)$$

It is then straightforward to deduce that no Killing spinor can be found for ϕ that satisfies the equation of motion.

It is worth remarking that the integrability condition (41) can be satisfied without imposing $\Gamma_-\epsilon = 0$. This requires that

$$H = \frac{1}{r^2} + y^2. \quad (45)$$

The resulting metric is given by

$$ds^2 = -dt^2 + r^2(-y^2dt^2 - 2dxdt + dy^2) + \frac{dr^2}{r^2}. \quad (46)$$

Note that this metric has also the scaling symmetry (2) with $z = 0$. In fact it is easy to verify that this metric is maximally symmetric and hence locally AdS.

After some straightforward calculation, we find that the Killing spinors that are independent of the null coordinate x are given by

$$\epsilon = \left(1 - \frac{1}{2}yr\Gamma_y(\Gamma_r + 1)\right)\left(2\sqrt{r} - \sqrt{\frac{1}{r}}\cot t\Gamma_-\right)\left(\eta_1 + \sin t\eta_2\right), \quad (47)$$

where η_1 and η_2 are constant spinors satisfying the following projection

$$\Gamma_-\eta_1 = 0 = \Gamma_+\eta_2, \quad (\Gamma_r + 1)\eta_1 = 0 = (\Gamma_r + 1)\eta_2. \quad (48)$$

Note that neither $\Gamma_-\epsilon$ nor $(\Gamma_{\bar{r}} + 1)\epsilon$ vanishes, but instead $\Gamma_-(\Gamma_{\bar{r}} + 1)\epsilon = 0$. It is easy to verify that

$$(\Gamma_{\bar{r}} + 1)\epsilon = -\frac{1}{2r}\cot t\Gamma_-\epsilon. \quad (49)$$

This type of Killing spinors was obtained in [24] in pp-wave solutions with enhanced supersymmetry. The remaining two Killing spinors are dependent on x . It is a linear function of x since it is easy to verify that $\partial_x^2\epsilon = 0$. In applying the AdS/CFT correspondence for Schrödinger solution, the null coordinate is typically treated as periodic and its quantized momentum lifts the mass of a free scalar mode [4, 5]. For the metric (46), treating the coordinate x periodic breaks half of the supersymmetry.

5 Conclusion

In this paper, we have obtained the Schrödinger solutions in Einstein-Weyl Supergravity. There are two types of such solutions: the ones which are pure gravitational and those which are supported by the massive vector. The exponent z of the Schrödinger solutions is determined by the product of the cosmological constant and the coupling of the Weyl-square super invariant term. We then demonstrate that the pure gravitational Schrödinger solutions are supersymmetric, preserving $\frac{1}{4}$ of the supersymmetry. The solutions with non-vanishing massive vector breaks all the supersymmetry.

We then obtain the general pp-wave solutions in Einstein-Weyl supergravity, which contain the Schrödinger metrics as special solutions. Furthermore, we have examples that describe flows from one Schrödinger vacuum to another. We show that the pure gravitational pp-waves all preserve $\frac{1}{4}$ of the supersymmetry, whilst turning on the massive vector field breaks all the supersymmetry. This is quite opposite from the Lifshitz solutions in Einstein-Weyl supergravity. It was shown that the supersymmetric Lifshitz solutions in Einstein-Weyl supergravity are all supported by the massive vector.

Higher derivative gravities and supergravities have rich structures for constructing geometric backgrounds that are dual to both relativistic and non-relativistic field theories. This work shows that non-relativistic field theories from Schrödinger backgrounds can be studied in the context of supersymmetry.

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